Indag. Mathem., N.S., 12 (3), 293–302 September 24, 2001

Complex projective foliations having sub-exponential growth

by B. Azevedo Scárdua

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, 21945-970 Rio de Janeiro – RJ, Brazil
e-mail: scardua@dmm.im.ufrj.br

Communicated by Prof. L.A. Pelletier at the meeting of May 28, 2001

ABSTRACT

Let X be a polynomial vector field in $\mathbb{C}^2$ and denote by $\mathcal{F}$ the corresponding holomorphic foliation in $\mathbb{C}P(2)$. Assume that (i) $\mathcal{F}$ has hyperbolic singularities; (ii) for some Riemannian metric on $\mathbb{C}P(2)$, hermitian along the leaves of $\mathcal{F}$, these leaves have sub-exponential growth. Then $\mathcal{F}$ is a hyperbolic linear foliation. In particular, the limit set of $\mathcal{F}$ is a union of singularities and invariant algebraic curves. It is interesting to regard this result under the standpoint of [16].

1. INTRODUCTION

It is well-known that the qualitative results for ordinary differential equations in the plane $\mathbb{R}^2$, as e.g. Poincaré-Bendixson Theorem, do neither generalize trivially to flows in higher dimension, nor to codimension-one foliations on arbitrary manifolds. It was J.F. Plante who pointed out the necessity of making growth restrictions on the leaves in order to obtain, for real codimension-one foliations, such generalizations ([16], [17]). The main idea is that the leaves must fulfill a sub-exponential growth condition, which implies the existence of holonomy invariant transverse measures. This condition is clearly verified for real flows in dimension 2 [16], and the consequences of the existence of such measures in the holonomy pseudogroups give then the appropriated tools for the study of such foliations.

In the present paper our aim is to study the situation above in the context of complex differential equations. More precisely, for holomorphic codimension-one foliations with singularities. The case of foliations by curves in a compact complex surface seems to be the first to be understood.

Let X be a polynomial vector field on the affine space $\mathbb{C}^2$. Since X is algebraic its local flow induces a singular (holomorphic) foliation by curves $\mathcal{F}$ on the projective space $\mathbb{C}P(2)$, and any foliation by curves on $\mathbb{C}P(2)$ is obtained this way. Thus, we regard foliations by curves on the complex projective plane $\mathbb{C}P(2)$, and consider the following question:

Question 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ and assume that for some C^∞ Riemannian metric g on $\mathbb{C}P(2)$, hermitian along the leaves of $\mathcal{F}$, these leaves exhibit a sub-exponential growth. Then, what can be said about $\mathcal{F}$?*

We refer to [16],[17] and [8] for the notions of growth of a leaf and of an holonomy pseudogroup that we will use below. Also we remark that if we take g as the Fubini-Study metric on $\mathbb{C}P(2)$, then it is easily checked that a *hyperbolic linear foliation* $\mathcal{L}$ on $\mathbb{C}P(2)$ say, $\mathcal{L}: \dot{x} = x, \dot{y} = \lambda y, \lambda \in \mathbb{C} \setminus \mathbb{R}$, in some affine chart $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$, has all its leaves with sub-exponential growth. On the other hand, if we take any rational map $R: \mathbb{C}P(2) \to \mathbb{C}P(2)$, then the pull-back foliation $\mathcal{F}^* = R^*(\mathcal{L})$, also has all its leaves with sub-exponential growth, but, in the case R is not an automorphism, $\mathcal{F}^*$ must have some degenerate or some non-hyperbolic singularity. Thus, for simplicity, we will assume that the singularities of $\mathcal{F}$ are *hyperbolic* which means that for each singularity $p \in \operatorname{sing} \mathcal{F} \subset \mathbb{C}P(2)$,of the foliation with small 3-spheres centered at the singularity. Our main result is:

Theorem 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$, with hyperbolic singularities. Assume $\mathcal{F}$ exhibits some leaf L_0 which is non-algebraic and has sub-exponential growth, for some C^∞ metric g on $\mathbb{C}P(2)$, hermitian along the leaves. Then $\mathcal{F}$ is linear hyperbolic of the form $\mathcal{F}|_{\mathbb{C}^2}: \dot{x} = x, \dot{y} = \lambda y$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, for some affine space $\mathbb{C}^2 \subset \mathbb{C}P(2)$.*

By an *algebraic leaf* we mean a leaf L whose closure $\bar{L} \subset \mathbb{C}P(2)$ is an algebraic (invariant) curve. We remark that according to the Index Theorem [4],[13] any algebraic invariant curve must have some singularity of the foliation. Moreover, any analytic subset of $\mathbb{C}P(2)$ is algebraic. Therefore, in a certain sense, *algebraic leaves play the role of compact leaves.*

Sketch of the proof

The basic idea is the following: by doubling the foliation we may use the sub-exponential growth and construct an holonomy invariant measure [17]. This measure will be supported in an algebraic leaf $\Lambda \subset \mathbb{C}P(2)$ by the fact that (in this case) there are no exceptional minimal sets supporting invariant measures for foliations in $\mathbb{C}P(2)$ [7]. Then the idea is to show that the holonomy group of this algebraic leaf must be solvable. The strategy is the following: first we use the more general statements in [16] (for non-compact foliated manifolds) and the geometry of hyperbolic foliations on $\mathbb{C}P(2)$ in order to prove the existence of a non-trivial invariant measure μ for the restriction $\mathcal{F}|_{\mathbb{C}P(2)\setminus\Lambda}$ (see Remark 1 below for what occurs in the linear case). Using now the fact that non-solvable groups of germs of one variable complex diffeomorphisms exhibit dense orbits and dense set of hyperbolic fixed points (in the sense of [15],[2],[21]) we may conclude that, in the non-solvable case, this measure would be at the same time atomic and with dense support in $\mathbb{C}P(2)\setminus\Lambda$. This contradiction gives the solvability of the holonomy groups of the irreducible components of Λ. This solvability implies that the foliation is transversely affine outside some algebraic invariant set of dimension one and therefore it must be a logarithmic foliation on $\mathbb{C}P(2)$ [18]. Since $\mathcal{F}$ has hyperbolic singularities it follows that it is linear in some affine chart.

Remark 1. For a linear hyperbolic foliation $\mathcal{F}$ on $\mathbb{C}P(2)$, given in affine coordinates by $xdy-\lambda ydx=0$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, we have three invariant projective lines (the coordinate axis and the line at the infinity) and no other algebraic leaves. The union Λ of these lines has components with linearizable holonomy groups and, since $\mathcal{F}$ is given in $\mathbb{C}P(2)\setminus\Lambda$ by the closed holomorphic 1-form $\omega=(dy/y)-\lambda(dx/x)$, the restriction $\mathcal{F}|_{\mathbb{C}P(2)\setminus\Lambda}$ has trivial holonomy and (therefore) does exhibit invariant transverse measures. Interesting metric results for foliations by curves on $\mathbb{C}P(n)$ can be found in [6],[12].

This work was partially supported by CNPq-Brasil.

2. PRELIMINARIES

Let (L,g) be a connected oriented Riemannian manifold of class $C^r, r\geqslant 1$. Given any point $x\in L$ the *growth function* of L at x is defined by $\gamma_x(r):=$ volume of the closed metric ball $B_L[x,r]$ of center x and radius r. The growth type of γ_x does not depend on the choice of $x\in L$. This way we may introduce the noltion of *polynomial growth, exponential growth, ...* for (M,g). If M is compact then it has polynomial growth of degree zero (see [8] Chp.V for details).

Let now (M,g) be a Riemannian manifold, perhaps non-compact, and let $\mathcal{F}$ be a (regular) C^1 foliation of codimension k on M. Assume that M is oriented and $\mathcal{F}$ is transversely oriented (this is always the case for holomorphic foliations). For each $x\in M$ denote by L_x the leaf of $\mathcal{F}$ through x. The metric on M induces a metric g_x along the (immersed) Riemannian manifold L_x.

Definition 1. The *growth type* of the leaf L_x with respect to the metric g is the growth type of the Riemmanian manifold (L_x,g_x).

Therefore, compact leaves have polynomial growth of degree zero. Let now $\mathcal{F}$ be a foliation with hyperbolic singularities on $\mathbb{C}P(2)$, and let g be a C^∞ Riemannian metric, hermitian along the leaves of $\mathcal{F}$. Write sing $(\mathcal{F})=\{p_1,\ldots,p_r\}$ and for each $j\in\{1,\ldots,r\}$ choose a small closed ball centered at p_j say, $\bar{\mathbb{B}}(p_j)\ni p_j$. Since p_j is hyperbolic $\mathcal{F}$ will be transverse to the spheres $\partial\{\bar{\mathbb{B}}(p_j)$ if they are small enough, as one can easily verify. Thus we may consider the double of $\mathcal{F}$ as in [3],[19]. This is a C^∞ regular codimension-two real foliation $\mathcal{F}_d$ on a compact real 4-manifold M_d, which contains two copies of the foliated pair $(\mathbb{C}P(2)\bigcup_{j=1}^{r}\bar{\mathbb{B}}(p_j), \mathcal{F}|_{\mathbb{C}P(2)\setminus\cup_{j=1}^{r}}\bar{\mathbb{B}}(p_j))$. By Schwarz Reflection Principle [1] the leaves of $\mathcal{F}_d$ have also natural structures of Riemann surfaces. The metric g induces a C^∞ Riemannian metric g_d in M_d, that can be chosen to be hermitian along the leaves of $\mathcal{F}_d$.

Proposition 1. *Let L be a leaf of $\mathcal{F}$ with sub-exponential growth for g. The corresponding leaf L_d of $\mathcal{F}_d$ has sub-exponential growth for g_d.*

The proof is based in the following remarks:

Lemma 1. *Let (M, g) be a (complete) Riemannian manifold and $B[p, R] \subset M$ a compact metric ball in M. Put $N = M \setminus B[p, r]$. Given any $x \in N$ we denote by $\gamma_{M,x}(r)$ the growth function of M and by $\gamma_{N,x}(r)$ the growth function of N at x. Then the growth type of $\gamma_{N,x}$ is not greater then the growth type of $\gamma_{M,x}$, indeed we have $\gamma_{N,x}(r) \leqslant \gamma_{M,x}(r)$, $\forall r \geqslant 0$.*

Proof. Denote by d_M and d_N the distance functions induced in M and N respectively. For any $x, y \in N$ it is clear that $d_M(x, y) \leqslant d_N(x, y)$. Therefore, we have $B_N[x, r] \subseteq B_M[x, r]$, $\forall x \in N, \forall r \geqslant 0$. Finally, since N is an open subset of M we have that the volume elements induced by the metric in M and N satisfy $\mathrm{Vol}_N = (\mathrm{Vol}_M)\big|_N$. Therefore the above inclusion gives $\mathrm{Vol}_N(B_N[x, r]) \leqslant \mathrm{Vol}_M(B_M[x, r])$. $\square$

Lemma 2. *Let (L, g) be a Riemannian manifold with non-empty boundary ∂L. Denote by $L_d = L \cup_{Id} L$ the manifold obtained by glueing two copies of L through the boundary by the identity diffeomorphism $Id\colon \partial L \to \partial L$. Endow L_d with the natural Riemannian metric g_d inherited from (L, g). Then the growth type of (L_d, g_d) is not greater then the growth type of (L, g).*

Proof. Let us denote by L_1, L_2 the two copies of L in L_d. Given any point $x_1 \in L_1$ we will denote by $x_2 \in L_2$ corresponding point in the copy L_2. Let us denote distances induced by the metrics on L_d, L_1, L_2 by ρ_d, ρ_1, ρ_2 respectively. Take any point $y \in L_d$ such that $\rho_d(x_1, y) \leqslant r$. Let $0 < a = \rho_d(x_1, x_2)$. We have two basic cases two consider:

Case 1: $y \in L_1$. In this case we have $\rho_d(x_1, y) = \rho_1(x_1, y)$ so that $\rho_1(x_1, y) \leqslant r$.

Case 2: $y \in L_2$. In this case, $\rho_2(x_2, y) = \rho_d(x_2, y) \leqslant \rho_d(x_2, x_1) + \rho_d(x_1, y) \leqslant r + a$.

Therefore we obtain the following inclusion of closed metric balls: $B_d[x_1, r] \subset B_1[x_1, r] \cup B_2[x_2, a + r]$. This inclusion proves the lemma. $\square$

Proposition 1 follows from Lemmas 1 and 2. Using Proposition 1 and Theorem 4.1 of [17] we obtain:

Proposition 2. *Let $\mathcal{F}, g, L_o$ be as in Theorem 1. Then there exists a non-trivial holonomy invariant transverse measure μ for $\mathcal{F}_d$, which is finite on compact sets and whose support is contained in the closure $\bar{L}_d$ of some leaf L_d of $\mathcal{F}_d$.*

Remark 2. Alternatively, instead of using the above argumentation based on the construction of the double, one may prove the existence of the holonomy invariant measure μ by observing that $M = \mathbb{C}P(2) \setminus \bigcup_{j=1}^{r} \mathbb{B}(p_j)$ is a compact manifold with boundary and $\mathcal{F}|_M$ is a foliation transverse to the boundary of M. Moreover, also one may observe that as a consequence of the compactness of $\mathbb{C}P(2)$ and of the Maximum Principle, $\mathcal{F}|_{\mathbb{C}P(2)\setminus \text{sing}\,\mathcal{F}}$ admits a *compact total transverse section*, that is, a compact (real C^∞) 2-manifold $\Sigma \subset \mathbb{C}P(2) \setminus \text{sing}\,\mathcal{F}$ with boundary, such that every leaf of $\mathcal{F}$ intersects the interior of Σ (indeed, Σ is a finite union of closed disks whose interiors are transverse to $\mathcal{F}$, see [5]). Therefore, according to the Remark after Corollary 4.2 in page 340 of [17], it follows that $\mathcal{F}$ admits a holonomy invariant transverse measure.

Now we take $K = \text{supp}(\mu) \subset M_d$ the support of μ on M_d, so that we have a compact $\mathcal{F}_d$-saturated non-empty set.

Lemma 3. *$K \cap \partial\mathbb{B}(p_j) \neq \emptyset$ for some $j \in \{1, ..., r\}$.*

Proof. In fact, otherwise we would have a non-trivial minimal set on $\mathbb{C}P(2)$, supporting an invariant transverse measure, what is not possible by [7]. □

Thus we may consider some intersection $K \cap \mathbb{B}(p_j) = K_j \neq \emptyset$. This is a compact invariant set supporting the induced invariant measure $\mu_j = \mu|_{\partial B(p_j)}$ for the induced transversely holomorphic flow $\mathcal{L}_j := \mathcal{F}|_{\partial B(p_j)}$. The flow $\mathcal{L}_j$ has two periodic orbits, which are hyperbolic and this implies that μ_j is concentrated in these orbits. Thus $K \cap \partial B(p_j)$ is contained in the union $\{xy = 0\}$ of the two local separatrices of $\mathcal{F}$ through p_j, given by the local form $\dot{x} = x, \dot{y} = \lambda_{p_j} y$. This implies that if L is a leaf of $\mathcal{F}$ such that $L \setminus \bigcup_{j=1}^{r} \bar{\mathbb{B}}(p_j) \subset K$, then $\bar{L} = L \cup \{p_{j_1}, ..., p_{j_t}\}$ for some $p_{j_\nu} \in \text{sing}\,\mathcal{F}$, and therefore by Remmert-Stein Theorem and Chow Theorem [9], $\bar{L}$ is an algebraic curve on $\mathbb{C}P(2)$. We conclude that:

Lemma 4. *Let $\mathcal{F}, g$ be as in Theorem 1. Then $\mathcal{F}$ has some algebraic invariant curve $\Lambda \subset \mathbb{C}P(2)$.*

3. SOLVABLE HOLONOMY GROUPS

From now on we will consider the following situation: $\mathcal{F}$ is a foliation with hyperbolic singularities in $\mathbb{C}P(2)$, and sub-exponential growth of some non-algebraic leaf L_o, for some C^∞ Riemannian metric g hermitian along the leaves. Denote by $\text{Diff}(\mathbb{C})$ the group of germs of complex diffeomorphisms fixing the origin $0 \in \mathbb{C}$. We consider the the reunion $\Lambda \subset \mathbb{C}P(2)$ of all the algebraic leaves of $\mathcal{F}$. According to Lemma 4 we have $\Lambda \neq \emptyset$. Notice that Λ has a finite number of irreducible components, otherwise by a Theorem of Darboux [11] $\mathcal{F}$ would admit a rational first integral, what is not compatible with the hyperbolic singularities. Therefore, Λ is an algebraic curve with nodal singularities appearing necessarily at singularities of $\mathcal{F}$. Fix an irreducible component Λ_1 of Λ, a point $q_1 \in \Lambda_1 \setminus \operatorname{sing} \mathcal{F}$, and a small transverse disk $\mathbb{D} \approx D_1 \subset \mathbb{C}P(2)$, with $D_1 \cap \Lambda = \{q_1\}$. We may consider the holonomy group $\operatorname{Hol}(\mathcal{F}, \Lambda_1, D_1)$ as a subgroup of $\operatorname{Diff}(D_1, q_1) \approx \operatorname{Diff}(\mathbb{C}, 0)$ [14].

Proposition 3. *Preserve $\mathcal{F}, \Lambda_1, L_o$ as above. The holonomy group* $\operatorname{Hol}(\mathcal{F}, \Lambda_1, D_1)$ *is a solvable group.*

In the proof of this proposition we will need a few considerations concerning non-solvable subgroups of germs of complex diffeomorphisms. Let $G \subset \operatorname{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup.

Theorem 2. [[2],[15],[21]]. *Suppose G is non-solvable.*

(i) *The basin of attraction of (the pseudo-orbits of) G is an open neighborhood of the origin $0 \in \Omega$.*

(ii) *Either G has dense pseudo-orbits in some neighborhood V of the origin or there exists an invariant germ of analytic curve Γ (equivalent to $\{\operatorname{Im.}(z^k) = 0\}$ for some $k \in \mathbb{N}$) where G has dense pseudo-orbits and also G has dense pseudo-orbits in each component of $V \setminus \Gamma$.*

(iii) *There exists a neighborhood $0 \in V \subset \Omega$, where G has a dense set of hyperbolic fixed points.*

Using Theorem 2 (iii) we obtain:

Corollary 1. *If a subgroup $G < \operatorname{Diff}(\mathbb{C}, 0)$ preserves some invariant measure, which is not the Dirac measure with atom at the origin, then G is solvable.*

The techniques and features developed above may be useful in a more general situation than the compact case.

Definition 2. Let $\mathcal{F}$ be a foliation of codimension k on a manifold M (perhaps non-compact). A *compact total transverse section* ot $\mathcal{F}$ is a compact k-manifold $\Sigma \subset M$ (possibly with boundary) such that every leaf of $\mathcal{F}$ intersects the interior of Σ.

The existence of holonomy invariant measure is also valid in this case (see Remark 2 and [17] page 340):

Theorem 3. [Plante, [17]]. *$\mathcal{F}$ be a C^2 foliation of codimension $k \geqslant 1$ on the Riemannian manifold (M, g). Assume that: (i) $\mathcal{F}$ exhibits a leaf L having subexponential growth with respect to the induced metric. (ii) $\mathcal{F}$ admits a compact total transverse section. Then there exists a non-trivial holonomy invariant mea-*

sure μ for $\mathcal{F}$ which is finite on compact sets and which has support contained in the closure $\overline{L} \subset M$ of L.

The special geometry of hyperbolic foliations on $\mathbb{C}P(2)$ gives us the following:

Lemma 5. *Let $\mathcal{F}$ be a foliation with hyperbolic singularities on $\mathbb{C}P(2)$ and denote by Λ its set of algebraic leaves. Then the restriction $\mathcal{F}|_M$, where $M = \mathbb{C}P(2) \setminus \Lambda$, admits a compact total transverse section.*

Proof. We may write $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_r$ in irreducible components. Take small transverse disks $\mathbb{D} \approx D_j \subset \mathbb{C}P(2)$ such that $D_j \cap \Lambda_j = \{q_j\} \subset \Lambda_j \setminus \operatorname{sing} \mathcal{F}, j = 1, \ldots, r$. Each component Λ_j must contain some hyperbolic singularity and therefore each holonomy group $\operatorname{Hol}(\Lambda_j, D_j, q_j)$ of the corresponding leaf must contain some hyperbolic element f_j. Using the fact that such an element f_j is linearizable and has linear part $|f_j'(0)| \neq 1$ we can construct a fundamental domain $A_j \subset D_j$ which is diffeomorphic to an annullus and such that each leaf L that accumulates some *regular* point in $\Lambda_j \setminus \operatorname{sing} \mathcal{F}$, necessarily intersects $D_j \setminus \{q_j\}$, accumulates q_j in D_j and therefore intersects the interior of the domain A_j. Let now $x \in M = \mathbb{C}P(2) \setminus \Lambda$ be a regular point and denote by L_x the corresponding leaf of $\mathcal{F}$ in M.

Claim 1. *L_x accumulates any regular point $q \in \Lambda \setminus (\operatorname{sing} \mathcal{F} \cap \Lambda)$.*

Proof (Claim 1). Indeed, it is well-known that $\mathbb{C}P(2) \setminus \Lambda$ is a Stein manifold [20]. Since $\overline{L_x}$ is non-algebraic this implies that L_x must accumulate Λ at some point. If it accumulates some singularity $q \in \Lambda \cap \operatorname{sing} \mathcal{F}$ then since this singularity is hyperbolic it follows from the local picture of hyperbolic singularities that either L_x is contained in the set of local separatrices of $\mathcal{F}$ through q or L_x accumulates both separatrices, and in this case (since Λ is invariant) L_x must accumulate regular points in Λ. Therefore, if L_x accumulates no regular point in Λ then we conclude that the closure $\overline{L_x}$ is in a neighborhood of Λ is analytic of pure dimension one. Since $\mathbb{C}P(2) \setminus \Lambda$ is a Stein manifold the Extension Theorem of Levi [20] applies to show that $\overline{L_x}$ is analytic of pure dimension one on $\mathbb{C}P(2)$ and again by Chow's Theorem [9] $\overline{L_x}$ will be an algebraic curve, contradiction. Therefore L_x accumulates some component Λ_{j_o} and by the local behaviour of hyperbolic singularities L_x must accumulate both local separatrices of $\mathcal{F}$ at any singularity $p \in \Lambda_{j_o}$, and since Λ is connected, it follows that L_x accumulates every component Λ_j also at regular points. This proves Claim 1. $\square$

In particular, $L_x \cap D_j$ accumulates q_j for each disk D_j. Using the above argumentation we conclude that L_x intersects some interior of domain A_j. This proves that $\mathcal{F}|_M$ admits a compact transverse section which is a finite union of domains A_j. Lemma 5 is now proved. $\square$

Proof of Proposition 3. According to Lemma 5 and Theorem 3 the foliation $\mathcal{F}|_{\mathbb{C}P(2)\setminus\Lambda}$ has a non-trivial holonomy invariant measure μ such that $\operatorname{supp}(\mu) \subset$

$\overline{L} \subset \mathbb{C}P(2) \setminus \Lambda$. Assume by contradiction that $\mathrm{Hol}(\mathcal{F}, \Lambda_1)$ is non-solvable. Preserve the transverse disk $\mathbb{D} \approx D_1$ to $\mathcal{F}$ with $\{q_1\} = D_1 \Lambda_1$ so that $\mathrm{Hol}(\Lambda_1, D_1, q_1) \subset \mathrm{Diff}(D_1, q_1) \cong \mathrm{Diff}(\mathbb{C}, 0)$ is non-solvable. We already know that the non-algebraic leaf L_o must accumulate the point q_1 in D_1 and therefore as it follows from Theorem 2, the leaf L_o is dense in a neighborhood $V \subset D_1$ of q_1 and $\mathrm{Hol}(\mathcal{F}, \Lambda_1, D_1)$ has a dense set of hyperbolic periodic pseudo-orbits on V (i.e., a dense set of hyperbolic fixed points). Given such a fixed point $z_o \in V$ we consider the word $f \in \mathrm{Hol}(\mathcal{F}, \Lambda_1, D_1)$ such that $f(z_o) = z_o$, with $|f'(z_o)| \neq 1$. This map f may be linearized in a small disk $D(z_o, \epsilon) \subset V$ as $f(z) = \lambda . z$ with $\lambda \in \mathbb{C}^* \setminus S^1$. Notice that this local map corresponds to the local holonomy map of some cycle γ_{z_o} in the leaf $L_{z_o} \subset M$ and therefore, since μ is $\mathrm{Hol}(\mathcal{F}|_M, L_{z_o})$-invariant it follows that $\mu|_{D(z_o,\epsilon)}$ is the Dirac measure with atom at z_o. In particular, $\mathrm{supp}(\mu) \cap D(z_o, \epsilon) = \{z_o\}$. On the other hand, since L_x is dense in V we have $\overline{L_x} \cap D(z_o, \epsilon) = D(z_o, \epsilon)$ which gives $\mathrm{supp}(\mu) \cap D(z_o, \epsilon) = D(z_o, \epsilon)$, contradiction. This contradiction proves Proposition 3. □

4. PROOF OF THEOREM 1

We shall now finish the proof of our main result:

Proof of Theorem 1. We may consider an affine chart $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$, and a polynomial 1-form $\omega = Pdx + Qdy$ with isolated singularities, which defines $\mathcal{F}|_{\mathbb{C}^2}$ and assume that $\mathbb{C}P(2) \setminus \mathbb{C}^2$ is not invariant. Let $\Lambda_1 \subset \mathbb{C}P(2)$ be any irreducible component of Λ as above. Since the corresponding holonomy group $\mathrm{Hol}(\mathcal{F}, \Lambda_1)$ is solvable, and $\mathcal{F}$ has hyperbolic singularities, it follows from [18] (see Proposition 5.1 page 185) that there exists a closed rational 1-form η on $\mathbb{C}P(2)$, with simple poles, such that $d\omega = \eta \wedge \omega$. This form is mainly constructed as follows (one can also follow [19], §§4 and 5). Fix any point $q_1 \in \Lambda_1 \setminus \mathrm{sing}\,(\mathcal{F})$ and a local transverse disk D_1 with $D_1 \cap \Lambda_1 = \{q_1\}$. A result of Cerveau and Moussu implies that there exists a local holomorphic coordinate $y \colon (D_1, q_1) \to (\mathbb{C}, 0)$ such that each element $h \in \mathrm{Hol}(\mathcal{F}, \Lambda_1, D_1)$ is of the form $h(y) = \lambda_h y/(1 + a_h y^k)^{1/k}$ for some $\lambda_h \in \mathbb{C}^*$, $a_h \in \mathbb{C}$ (cf. [18] page 185). In other words, the holonomy group is a finite covering of a group of germs of affine maps. In a neighborhood of q_1 we may write $\omega = gdy$ for some meromorphic function g. The one form η is then given in this neighborhood by $\eta = (k+1)(dy/y) + (dg/g)$. These local models glue, because two coordinates y and $\tilde{y}$ as above must be related by a relation of the form $\tilde{y} = \lambda y/(1 + a y^k)^{1/k}$ in the common domain. Thus, η is well-defined in a neighborhood of $\Lambda_1 \setminus \mathrm{sing}\,(\mathcal{F})$, it also extends meromorphically to a neighborhood of Λ_1 on $\mathbb{C}P(2)$ because of the hyperbolicity of the singularities ([18] Lemma 3.2 page 177). Finally, η extends to $\mathbb{C}P(2)$ by the Extension Theorem of Levi ([18] Remark 4.1 page 180). In particular, $\mathcal{F}$ is transversely affine in $\mathbb{C}P(2) \setminus \Lambda$ ([18] Theorem 4.1 page 178). Since $\mathcal{F}$ has hyperbolic singularities this implies that $\mathcal{F}$ is a Darboux (logarithmic) foliation ([18] Theorem 4.3 page 183). Now, according to a remark in [3], a logarithmic foliation whose all singularities on $\mathbb{C}P(2)$ are hyperbolic, must be linear as in Theorem 1. □

As a corollary of the proof given above, we obtain:

Corollary 2. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$, with hyperbolic singularities and having infinitely many leaves with sub-exponential growth for the Fubini-Study metric on $\mathbb{C}P(2)$. Then $\mathcal{F}$ is linear hyperbolic in some affine chart.*

Proof. Indeed, it is enough to observe that $\mathcal{F}$ has a finite number of algebraic leaves (see the beggining of section §3). Therefore, $\mathcal{F}$ exhibits some leaf L_o like in the statement of Theorem 1. $\square$

Corollary 3. *Let $\mathcal{F}$ be a foliation with hyperbolic singularities on $\mathbb{C}P(2)$ having $\Lambda \subset \mathbb{C}P(2)$ as the (nonempty) reunion of all the algebraic leaves. Assume that $\mathcal{F}|_{\mathbb{C}P(2)\setminus\Lambda}$ admits a non-trivial holonomy invariant measure on $\mathbb{C}P(2) \setminus \Lambda$. Then $\mathcal{F}$ is linear hyperbolic.*

Proof. It is enough to mimic the proof of Proposition 3 in order to show that any irreducible component of Λ has a solvable holonomy group and then proceed as in the proof of Theorem 1. $\square$

A comparison of the result in Theorem 1 with the results in [3] and [19] comes naturally to mind. We have therefore the following question:

Question 2. *Let $\mathcal{F}$ be a foliation by curves on $\mathbb{C}P(2)$. Is there any relation between the existence of parabolic leaves and the existence of leaves with sub-exponential growth for some suitable metric? What about a projective foliation whose leaves have polynomial growth? Are these foliations given by closed rational 1-forms?*

REFERENCES

1. Ahlfors, L. – Complex Analysis. MacGraw Hill, New York (1966).
2. Belliart, M., I. Liousse and F. Loray – Sur l'existence de points fixes attractifs pour les sous-groupes de Aut($\mathbb{C}$, 0). To appear in C.R. Acad. Sci. Paris Sér.I Math.
3. Brunella, M. – Foliations on the complex projective plane with many parabolic leaves. Ann. Inst. Fourier, Grenoble. **44** 4, 1237–1242 (1994).
4. Camacho, C. and P. Sad – Invariant varieties through singularities of holomorphic vector fields. Ann. of Math. **115**, 579–595 (1982).
5. Canille Martins, J.C. and B. Azevedo Scárdua – On the growth of holomorphic projective foliations. Pre-print (2000).
6. Candel, A. and X. Gómez-Mont – Uniformization of the leaves of a rational vector field. Ann. Inst. Fourier, Grenoble. **45**, 4, 1123–1133 (1995).
7. Camacho, C., A. Lins Neto and P. Sad – Minimal sets of foliations on complex projective spaces. Publ. Math. I.H.E.S., **68** 187–203.
8. Godbillon, C. – Feuilletages, Études Geométriques I, Université Louis Pasteur, Mai (1985).
9. Gunning, R. and H. Rossi – Analytic functions of several complex variables. Prentice Hall, Englewood Cliffs, NJ (1965).
10. Hirch-Smale, M. and S. Smale – Differential Equations, Dynamical Systems and Linear Algebra. Academic Press, New York and London (1970).
11. Jouanolou, J.P. – Équations de Pfaff algèbriques. Lecture Notes in Math. **708**, Springer-Verlag, Berlin (1979).

12. Lins Neto, A. – Simultaneous uniformization for the leaves of projective foliations by curves. Bol. Soc. Bras. Mat., Vol. **25**, N.2, 181–206 (1994).
13. Lins Neto, A. – Algebraic solutions of polynomial differential equations and foliations in dimension two. Lect. Notes in Math. no. 1345, 192–231.
14. Mattei, J.F. and R. Moussu – Holonomie et intégrales premières. Ann. Ec. Norm. Sup. **13**, 469–523 (1980).
15. Nakai, I. – Separatrices for nonsolvable dynamics on $\mathbb{C}, 0$. Ann. Inst. Fourier, Grenoble **44**, 2, pp. 569–599 (1994).
16. Plante, J.F. – A generalization of the Poincaré-Bendixson Theorem. Topology vol. **12**, 177–181 (1973).
17. Plante, J.F. – Foliations with measure preserving holonomy. Annals of Math., **102**, 327–361 (1975).
18. Azevedo Scárdua, B. – Transversely affine and transversely projective foliations. Ann. Sc. École Norm. Sup., 4[e] série, t.**30**, 169–204 (1997).
19. Azevedo Scárdua, B. – A Remark on Parabolic Projective Foliations. Hokkaido Mathematical Journal, Vol. **28-2**, 231–252 (1999).
20. Siu, Y. – Techniques of Extension of Analytic Objects. Marcel Dekker, N.Y. (1974).
21. Wirtz, B. – Entropy and fixed points for nonsolvable dynamics of $(\mathbb{C}, 0)$; Bol. Soc. Brasil. Mat. (N.S.) **29**, no. 1, 53–74 (1998).

(Received December 2000)